Birkhoff coordinates for the focusing NLS equation

Kappeler, T ; Lohrmann, P ; Topalov, P ; Zung, N T

Abstract: In this paper we construct Birkhoff coordinates for the focusing nonlinear Schrödinger equation near the zero solution.

DOI: https://doi.org/10.1007/s00220-008-0543-0

Originally published at:
Kappeler, T; Lohrmann, P; Topalov, P; Zung, N T (2009). Birkhoff coordinates for the focusing NLS equation. Communications in Mathematical Physics, 285(3):1087-1107.
DOI: https://doi.org/10.1007/s00220-008-0543-0

Commun. Math. Phys. 285, 1087–1107 (2009)
Digital Object Identifier (DOI) 10.1007/s00220-008-0543-0

Communications in **Mathematical Physics**

Birkhoff Coordinates for the Focusing NLS Equation

T. Kappeler[1,⋆], **P. Lohrmann**[1,⋆⋆], **P. Topalov**[2], **N. T. Zung**[3]

[1] Institut für Mathematik, Universität Zürich, Winterthurerstrasse 190, 8057 Zürich, Switzerland.
E-mail: tk@math.uzh.ch; philipp.lohrmann@math.uzh.ch
[2] Department of Mathematics, Northeastern University, Boston, MA 02115, USA.
E-mail: p.topalov@neu.edu
[3] Institut de Mathématiques de Toulouse, Université Paul Sabatier,
118 Rte de Narbonne, 31062 Toulouse, France. E-mail: tienzung@picard.ups-tlse.fr

Received: 6 December 2007 / Accepted: 24 February 2008
Published online: 4 July 2008 –

Abstract: In this paper we construct Birkhoff coordinates for the focusing nonlinear Schrödinger equation near the zero solution.

1. Introduction

Consider the *focusing* nonlinear Schrödinger equation (fNLS)

$$i\partial_t\psi = -\partial_x^2\psi - 2|\psi|^2\psi \tag{1.1}$$

with periodic boundary conditions, i.e. $\psi(x+1,t) = \psi(x,t)$ for $x, t \in \mathbb{R}$. The fNLS equation (1.1) is integrable and admits a Lax-pair formalism – see [14]. It can be written in Hamiltonian form as follows. Let $L^2 := L^2(\mathbb{T}, \mathbb{C})$ denote the Hilbert space of L^2-integrable complex-valued functions on the circle $\mathbb{T} := \mathbb{R}/\mathbb{Z}$ and let $\mathcal{L}^2 := L^2 \times L^2$. For C^1-functionals F and G introduce the Poisson bracket

$$\{F, G\}(\varphi) = i\int_0^1 \left(\partial_{\varphi_1}F\,\partial_{\varphi_2}G - \partial_{\varphi_2}F\,\partial_{\varphi_1}G\right)dx, \tag{1.2}$$

where $\varphi = (\varphi_1, \varphi_2)$ and $\partial_{\varphi_i}F$ denotes the L^2-gradient of F with respect to φ_i, $i = 1, 2$. The Hamiltonian system with Hamiltonian

$$\mathcal{H} \equiv \mathcal{H}(\varphi) := \int_0^1 (\partial_x\varphi_1\partial_x\varphi_2 + \varphi_1^2\varphi_2^2)dx \tag{1.3}$$

is then given by

$$\partial_t(\varphi_1, \varphi_2) = i(-\partial_{\varphi_2}\mathcal{H}, \partial_{\varphi_1}\mathcal{H}). \tag{1.4}$$

⋆Supported in part by the Swiss National Science Foundation, and the European Community through the FP6 Marie Curie RTN ENIGMA (MRTN-CT-2004-5652).
⋆⋆Supported by the Swiss National Science Foundation.

Equation (1.1) is obtained by restricting (1.4) to the invariant subspace

$$i\mathcal{L}^2_{\mathcal{R}} := \left\{(\varphi_1, \varphi_2) \in \mathcal{L}^2 \mid \varphi_1 = -\bar{\varphi}_2\right\}.$$

With $(\varphi_1, \varphi_2) = (\psi, -\bar{\psi})$ one has

$$\partial_t \psi = i\partial_{\bar{\psi}} \mathcal{H}_{\mathrm{f}} = i\partial_x^2 \psi + 2i|\psi|^2\psi, \tag{1.5}$$

where

$$\mathcal{H}_{\mathrm{f}}(\psi) = \int_0^1 (-\partial_x \psi \partial_x \bar{\psi} + \psi^2 \bar{\psi}^2) dx. \tag{1.6}$$

When restricting (1.4) to the invariant subspace

$$\mathcal{L}^2_{\mathcal{R}} := \left\{(\varphi_1, \varphi_2) \in \mathcal{L}^2 \mid \varphi_1 = \bar{\varphi}_2\right\}$$

of $\mathcal{L}^2$ one obtains the *defocusing* nonlinear Schrödinger equation (dNLS). With $(\varphi_1, \varphi_2) = (\psi, \bar{\psi})$ one has

$$\partial_t \psi = -i\partial_{\bar{\psi}} \mathcal{H}_{\mathrm{d}} = i\partial_x^2 \psi - 2i|\psi|^2\psi, \tag{1.7}$$

where

$$\mathcal{H}_{\mathrm{d}}(\psi) = \int_0^1 (\partial_x \psi \partial_x \bar{\psi} + \psi^2 \bar{\psi}^2) dx.$$

Equation (1.4) admits the Lax pair representation

$$\partial_t L(\varphi) = [A(\varphi), L(\varphi)], \tag{1.8}$$

where $\varphi = (\varphi_1, \varphi_2)$, $L = L(\varphi)$ is the Zakharov-Shabat operator (ZS operator)

$$L(\varphi) := i \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \partial_x + \begin{pmatrix} 0 & \varphi_1 \\ \varphi_2 & 0 \end{pmatrix} \tag{1.9}$$

and

$$A(\varphi) := i \begin{pmatrix} -2\partial_x^2 + \varphi_1\varphi_2 & -\partial_x\varphi_1 - 2\varphi_1\partial_x \\ \partial_x\varphi_2 + 2\varphi_2\partial_x & 2\partial_x^2 - \varphi_1\varphi_2 \end{pmatrix}.$$

Birkhoff normal form. The theory of normal forms of integrable (or near integrable) systems aims at representing such systems in coordinates which are particularly suited to integrate them as well as to study their (Hamiltonian) perturbations. The most simple case is arguably the normal form of such systems near an isolated equilibrium solution. It goes back to Birkhoff and is usually referred to as Birkhoff normal form.

Assume that the origin 0 of $\mathbb{R}^n \times \mathbb{R}^n$ is an isolated equilibrium of some Hamiltonian system with real analytic Hamiltonian H and standard symplectic structure. It means that 0 is an isolated singular point of the corresponding Hamiltonian vector field X_H. For simplicity, we assume that H admits an expansion of the form

$$H = \frac{1}{2}\sum_{i=1}^n \lambda_i (q_i^2 + p_i^2) + \cdots,$$

where $z = (q, p)$ denotes a point near $0 \in \mathbb{R}^n \times \mathbb{R}^n$ and the dots stand for terms of higher order in z. The real numbers $\lambda_1, \ldots, \lambda_n$ are referred to as the frequencies of the linearized system. They are said to be nonresonant up to order m, if

$$\sum_{i=1}^{n} k_i \lambda_i \neq 0 \text{ whenever } 1 \leq \sum_{i=1}^{n} |k_i| \leq m,$$

where $k_1, \ldots, k_n$ are arbitrary integers and $m \geq 1$. They are nonresonant if they are nonresonant up to any finite order. A Hamiltonian H is in Birkhoff normal form up to order m if it is of the form

$$H = N_2 + N_4 + \cdots + N_m + \cdots ,$$

where the N_k, $2 \leq k \leq m$, are homogenous polynomials of order k, which are actually functions of $q_k^2 + p_k^2$, $1 \leq k \leq n$, and where $\ldots$ stands for terms of order strictly greater than m. If this holds for any m, the Hamiltonian is said to be in Birkhoff normal form. Birkhoff showed that if the frequencies $\lambda_1, \ldots, \lambda_n$ are nonresonant up to order $m \geq 3$, then there exists an analytic canonical transformation $\Phi = id + \cdots$ near 0 such that

$$H \circ \Phi = N_2 + N_4 + \cdots + N_m + \cdots$$

is in Birkhoff normal form up to order m. If the frequencies $\lambda_1, \ldots, \lambda_n$ are nonresonant up to any order, then this normalization process can be carried to any order. The resulting symplectic transformation, however, is in general no longer convergent in any neighborhood of the origin and can only be given the meaning of a formal power series.

If some canonical transformation into Birkhoff normal form were convergent, then the resulting Hamiltonian would be integrable in a neighborhood of the origin, the integrals in involution being $q_1^2 + p_1^2, \ldots, q_n^2 + p_n^2$. It turns out that a certain converse is true. If a Hamiltonian with a nonresonant elliptic equilibrium admits n functionally independent integrals in involution, then the formal transformation into Birkhoff normal form is convergent, hence the Hamiltonian itself is integrable. Such a result was proven by Vey [13] and then improved by Ito [7] and Zung [15]. Note that the normalizing transformation is typically only defined in a neighborhood of the elliptic equilibrium. In case the transformation is defined on all of phase space, one refers to the Birkhoff coordinates as *global* Birkhoff coordinates.

In the last decade, normal form theory has been extended to Hamiltonian PDEs. In particular, Birkhoff normal forms of finite order have been studied for Hamiltonian PDEs and applied to obtain results on long time asymptotics for solutions near an equilibrium – see e.g. [2] and references therein. As in Hamiltonian systems of finite dimension, in the case of *integrable* PDEs one expects stronger results to hold. First results in this direction were obtained for the KdV equation and the defocusing nonlinear Schrödinger equation – see [8], respectively, [6].

Denote by $H^N \equiv H^N(\mathbb{T}, \mathbb{C})$ the Sobolev space of complex valued functions on the circle $\mathbb{T}$,

$$H^N(\mathbb{T}, \mathbb{C}) := \{\psi(x) = \sum_{k \in \mathbb{Z}} e^{2\pi i k x} \hat{\psi}(k) : \|\psi\|_N < \infty\},$$

where for $N \geq 0$,

$$\|\psi\|_N := \left(\sum_{k \in \mathbb{Z}} (1 + |k|)^{2N} |\hat{\psi}(k)|^2 \right)^{\frac{1}{2}},$$

and $\hat{\psi}(k) := \int_0^1 \psi(x)e^{-2\pi ikx}dx$, $k \in \mathbb{Z}$, denote the Fourier coefficients of ψ. Further let $l^2_{\mathbb{C}^2}$ be the Hilbert space

$$l^2_{\mathbb{C}^2} = l^2(\mathbb{Z}, \mathbb{C}) \times l^2(\mathbb{Z}, \mathbb{C}), \quad (x, y) = (x_k, y_k)_{k\in\mathbb{Z}}.$$

We endow $l^2_{\mathbb{C}^2}$ with the standard Poisson bracket for which $\{x_k, y_k\} = -\{y_k, x_k\} = 1$ for any $k \in \mathbb{Z}$ whereas all other brackets between coordinate functions vanish. It induces the standard Poisson brackets on the real subspaces

$$l^2_{\mathbb{R}^2} := l^2(\mathbb{Z}, \mathbb{R}) \times l^2(\mathbb{Z}, \mathbb{R}) \quad \text{and} \quad il^2_{\mathbb{R}^2} := l^2(\mathbb{Z}, i\mathbb{R}) \times l^2(\mathbb{Z}, i\mathbb{R}).$$

More generally, for any $N \geq 0$, introduce

$$l^2_N \equiv l^2_N(\mathbb{Z}, \mathbb{C}) := \{x = (x_j)_{j\in\mathbb{Z}} |\, x \in l^2(\mathbb{Z}, \mathbb{C}),\, \|x\|_N < \infty\},$$

where

$$\|x\|_N := \left(\sum_{j\in\mathbb{Z}} (1 + |j|)^{2N} |x_j|^2\right)^{\frac{1}{2}} < \infty.$$

The main result of this paper is the following

Theorem 1.1. *There exist a neighborhood W_f of $0 \in i\mathcal{L}^2_{\mathcal{R}}$, a neighborhood U_f of $0 \in il^2_{\mathbb{R}^2}$, and a map*

$$\Phi_f : W_f \to U_f$$

such that

(i) *Φ_f is 1–1, onto, bi-analytic and preserves the Poisson bracket.*
(ii) *The coordinates $(x_k, y_k)_{k\in\mathbb{Z}} = \Phi_f(\varphi)$ are Birkhoff coordinates for the focusing NLS equation, i.e. for $\varphi \in i\mathcal{L}^2_{\mathcal{R}} \cap (H^1 \times H^1)$, the Hamiltonian $\mathcal{H}_f \circ \Phi_f^{-1}$ depends only on the action variables $I_k = \frac{1}{2}(x_k^2 + y_k^2)$, $k \in \mathbb{Z}$.*
(iii) *For any $N \geq 0$, Φ_f maps $W_f \cap (H^N \times H^N)$ diffeomorphically onto $U_f \cap (l^2_N \times l^2_N)$.*

Remark 1.1. Statement (iii) of Theorem 1.1 remains valid if the Sobolev space H^N is replaced by the weighted Sobolev space H^ω with subexponential weight ω and, correspondingly, the sequence space l^2_N by the weighted sequence space l^2_ω – see [9].

Theorem 1.1 can be used to obtain a KAM-result for the focusing NLS equation of the type obtained in [5] for the defocusing NLS equation. In the case of fNLS it is valid in a neighborhood of 0 of the invariant subspace of $i\mathcal{L}^2_{\mathcal{R}} \cup (H^N \times H^N)$ consisting of odd potentials. It improves on the KAM theorem established in [10] and can be proved in the same way as the corresponding result in [5].

To prove Theorem 1.1 we use that the defocusing NLS equation admits global Birkhoff coordinates. More precisely, in [6] it is shown that there exists a real analytic canonical map $\Phi : \mathcal{L}^2_{\mathcal{R}} \to l^2_{\mathbb{R}^2}$ which associates to a potential φ in $\mathcal{L}^2_{\mathcal{R}}$ its Birkhoff coordinates $(x_k(\varphi), y_k(\varphi))_{k\in\mathbb{Z}}$. The map Φ extends analytically to a map $W \to l^2_{\mathbb{C}^2}$, defined on an open neighborhood W of $\mathcal{L}^2_{\mathcal{R}}$ in $\mathcal{L}^2$. In order to provide Birkhoff coordinates on a neighborhood of 0 for the *focusing* NLS-equation, we will show that there exists a neighborhood of 0 in $i\mathcal{L}^2_{\mathcal{R}} \cap W$ so that the restriction of Φ to this neighborhood has all the properties listed in Theorem 1.1. The main point consists in verifying that

$$\Phi(i\mathcal{L}^2_{\mathcal{R}} \cap W) \subset il^2_{\mathbb{R}^2}.$$

2. Set-up

In this section we introduce some more notations, recall several results needed in the sequel and establish some auxiliary results.

2.1. Spectral properties of $L(\varphi)$ and its discriminant. For $\varphi = (\varphi_1, \varphi_2) \in \mathcal{L}^2$, consider the ZS operator $L(\varphi)$, defined by (1.9). For any $\lambda \in \mathbb{C}$, let $M = M(x, \lambda, \varphi)$ denote the fundamental 2×2 matrix of $L(\varphi)$,

$$L(\varphi)M = \lambda M,$$

satisfying the initial condition $M(0, \lambda, \varphi) = \mathrm{Id}_{2\times 2}$. The entries of M are denoted by M_{ij}, $(1 \leq i, j \leq 2)$.

Periodic spectrum. Denote by $\mathrm{Spec}_{\mathrm{per}}(\varphi)$ the spectrum of the operator $L = L(\varphi)$ with domain

$$\mathrm{dom}_{\mathrm{per}}(L) := \{F \in H^1_{\mathrm{loc}} \times H^1_{\mathrm{loc}} |\ F(1) = \pm F(0)\}.$$

This spectrum coincides with the spectrum of the operator $L(\varphi)$ considered on $[0, 2]$ with periodic boundary conditions. The following proposition is well known – see e.g. [6], Prop. I.6.

Proposition 2.1. *For any $\varphi \in \mathcal{L}^2$, the set of periodic eigenvalues of $L(\varphi)$ (listed with multiplicities) consists of a sequence of pairs $(\lambda_k^-(\varphi), \lambda_k^+(\varphi))$, $\lambda_k^\pm(\varphi) \in \mathbb{C}$, satisfying*

$$\lambda_k^\pm(\varphi) = k\pi + l^2(k)$$

locally uniformly in φ, i.e. $(\lambda_k^\pm(\varphi) - k\pi)_{k\in\mathbb{Z}} \in l^2(\mathbb{Z}, \mathbb{C})$ and the sequences are locally uniformly bounded.

We say that two complex numbers a, b are *lexicographically ordered*, $a \preccurlyeq b$, if

$$[\mathrm{Re}(a) < \mathrm{Re}(b)] \text{ or } [\mathrm{Re}(a) = \mathrm{Re}(b) \text{ and } \mathrm{Im}(a) \leq \mathrm{Im}(b)].$$

Proposition 2.2. (i) *For $\varphi = (\psi, \bar{\psi}) \in \mathcal{L}^2_{\mathcal{R}}$, the periodic eigenvalues $(\lambda_k^\pm(\varphi))_{k\in\mathbb{Z}}$ are real. Moreover, they can be listed (with multiplicities) in such a way that*

$$\ldots \lambda_{k-1}^+ < \lambda_k^- \leq \lambda_k^+ < \lambda_{k+1}^- \cdots . \tag{2.1}$$

(ii) *For potentials $\varphi = (\psi, -\bar{\psi}) \in i\mathcal{L}^2_{\mathcal{R}}$ the periodic eigenvalues $(\lambda_k^\pm(\varphi))_{k\in\mathbb{Z}}$ can be listed (with multiplicities) in such a way that $Im(\lambda_k^+) \geq 0\ \forall k \in \mathbb{Z}$, and $(\lambda_k^+(\varphi))_{k\in\mathbb{Z}}$ is lexicographically ordered. In addition, for any $k \in \mathbb{Z}$, λ_k^- is given by*

$$\lambda_k^- = \overline{\lambda_k^+}.$$

Proof. (i) For $\varphi \in \mathcal{L}^2_{\mathcal{R}}$, the operator $L(\varphi)$ with periodic boundary conditions is self-adjoint, hence its spectrum is real. The sequence of inequalities (2.1) follows from [6], formula (I.20). (ii) The claimed statement follows from Proposition 2.1 and the fact that if $F = (F_1, F_2)$ is a periodic eigenfunction with eigenvalue λ, then $\check{F} := (-\bar{F}_2, \bar{F}_1)$ is a periodic eigenfunction with eigenvalue $\bar{\lambda}$. □

Dirichlet spectrum. For $\varphi \in \mathcal{L}^2$, denote by $\mathrm{Spec}_{\mathrm{dir}}(\varphi)$ the Dirichlet spectrum of the operator $L(\varphi)$, i.e. the spectrum of $L(\varphi)$ considered with domain

$$\mathrm{dom}_{\mathrm{dir}}(L) := \{F = (F_1, F_2) \in H^1([0,1], \mathbb{C})^2 \mid F_1(0) = F_2(0), F_1(1) = F_2(1)\}. \tag{2.2}$$

Note that the Dirichlet spectrum is discrete. The following results are well known – see e.g. [6] Prop. I.9, formula I.22.

Proposition 2.3. (i) *For $\varphi \in \mathcal{L}^2$ the Dirichlet eigenvalues $(\mu_k(\varphi))_{k \in \mathbb{Z}}$ can be listed (with multiplicities) in such a way that they are lexicographically ordered and satisfy the asymptotic estimates*

$$\mu_k(\varphi) = k\pi + l^2(k),$$

locally uniformly in φ.

(ii) *For $\varphi \in \mathcal{L}^2_{\mathcal{R}}$, the Dirichlet eigenvalues are real and satisfy*

$$\lambda_k^-(\varphi) \le \mu_k(\varphi) \le \lambda_k^+(\varphi).$$

Discriminant. Let $\Delta(\lambda, \varphi) := M_{11}(1, \lambda, \varphi) + M_{22}(1, \lambda, \varphi)$ be the trace of the fundamental matrix M evaluated at $x = 1$. It is well known that $\Delta(\lambda, \varphi)$ is an entire function on $\mathbb{C} \times \mathcal{L}^2$ (cf. [6], Lemma I.1). Denote by $\dot{\Delta}$ the partial derivative of $\Delta(\lambda, \varphi)$ with respect to λ. The following properties of $\Delta(\lambda, \varphi)$ are well known – see e.g. [11] or [6], Sect. I.2, Lemma I.19, Lemma I.20, and Lemma I.22.

Proposition 2.4. (i) *For any $\varphi \in \mathcal{L}^2$ and any $\lambda \in \mathbb{C}$,*

$$\Delta^2(\lambda, \varphi) - 4 = -4\left(\lambda_0^-(\varphi) - \lambda\right)\left(\lambda_0^+(\varphi) - \lambda\right) \prod_{k \neq 0} \frac{\left(\lambda_k^+(\varphi) - \lambda\right)\left(\lambda_k^-(\varphi) - \lambda\right)}{k^2\pi^2}.$$

(ii) *For any $\varphi \in \mathcal{L}^2$, the λ-derivative $\dot{\Delta}$ of $\Delta(\lambda, \varphi)$ has countably many roots. They can be listed (with multiplicities) in such a way that they are lexicographically ordered and satisfy the asymptotic estimates*

$$\dot{\lambda}_k = k\pi + l^2(k),$$

locally uniformly in φ. For any $\varphi \in \mathcal{L}^2$, $\dot{\Delta}(\lambda, \varphi)$ admits the following product representation:

$$\dot{\Delta}(\lambda, \varphi) = 2(\dot{\lambda}_0 - \lambda) \prod_{k \neq 0} \frac{\dot{\lambda}_k - \lambda}{k\pi}.$$

(iii) *For any $\varphi \in i\mathcal{L}^2_{\mathcal{R}}$ and $\lambda \in \mathbb{C}$,*

$$\Delta(\bar{\lambda}, \varphi) = \overline{\Delta(\lambda, \varphi)} \quad \textit{and} \quad \overline{\dot{\Delta}(\lambda, \varphi)} = \dot{\Delta}(\bar{\lambda}, \varphi).$$

Proof. The first two items are proved in [6], Sect. I.6. The third item is well known – see for example [1]. For the convenience of the reader we repeat the proof here. Let $F = F(x, \lambda, \varphi)$ be the solution of

$$L(\varphi)F = \lambda F \tag{2.3}$$

such that $F|_{x=0} = (1, 0)$. Then $F_i(x, \lambda, \varphi) = M_{i1}$ for $i = 1, 2$. A straightforward computation shows that

$$\check{F}(x, \lambda, \varphi) = \left(-\overline{F_2}(x, \bar{\lambda}, \varphi), \overline{F_1}(x, \bar{\lambda}, \varphi)\right)$$

is a solution of (2.3) with $\check{F}|_{x=0} = (0, 1)$. Hence, $\Delta(\lambda, \varphi) = F_1(1, \lambda, \varphi) + \overline{F_2}(1, \bar{\lambda}, \varphi)$. The latter equality proves the statement. □

Spectral properties of potentials in $i\mathcal{L}^2_{\mathcal{R}}$ *near* 0. Potentials $\varphi \in i\mathcal{L}^2_{\mathcal{R}}$ near the origin have additional spectral properties. To describe them let $(D_k)_{k\in\mathbb{Z}}$ denote the sequence of disks in $\mathbb{C}$ with center $k\pi$ and radius $\pi/4$.

Proposition 2.5. *There exists a neighborhood* W *of* 0 *in* $\mathcal{L}^2$*, such that, for any* $\varphi \in W \cap i\mathcal{L}^2_{\mathcal{R}}$ *and* $k \in \mathbb{Z}$*, the following properties hold:*

(i) $Spec_{per}(L(\varphi)) \cap D_k = \{\lambda_k^-, \lambda_k^+\}$;
(ii) $Crit(\Delta(\cdot, \varphi)) \cap D_k = \{\dot{\lambda}_k\}$;
(iii) $Spec_{dir}(L(\varphi)) \cap D_k = \{\mu_k\}$;
(iv) $\dot{\lambda}_k \in \mathbb{R}$, *and* $\Delta(\lambda_k^{\pm}(\varphi), \varphi) = 2(-1)^k$.

Proof. The existence of a neighborhood W of 0 in $\mathcal{L}^2$ so that any $\varphi \in W \cap i\mathcal{L}^2_{\mathcal{R}}$ satisfies items $(i) - (iii)$ follows from the fact that for $\varphi = (0, 0)$,

$$\lambda_k^- = \lambda_k^+ = \dot{\lambda}_k = \mu_k = k\pi \qquad \forall k \in \mathbb{Z}$$

together with Proposition 2.1, Proposition 2.3 (i) and Proposition 2.4 (ii).

By Proposition 2.4 (iv) the critical points $\dot{\lambda}_k$ of Δ are either real or they occur in complex conjugate pairs. By item (ii) they cannot occur in complex conjugate pairs. Hence they must be real. Further, by a deformation argument, one sees that $\Delta(\lambda_k^{\pm}(\varphi), \varphi) = 2(-1)^k$ and item (iv) is proved as well. □

2.2. Branches of the square root. We need to consider different branches of the square root.

Canonical branch. We denote by $\sqrt[+]{z}$ (or simply by $\sqrt{z}$) the principal branch of the square root defined on $\mathbb{C}\setminus\{x \in \mathbb{R} \mid x \le 0\}$ by $\sqrt[+]{1} = 1$. Given $a, b \in \mathbb{C}$ with $a \ne b$ and $a \preccurlyeq b$, we denote by $\sqrt[s]{(a-z)(b-z)}$ the standard branch of the square root, defined on $\mathbb{C}\setminus[a, b]$ and determined by

$$\sqrt[s]{(a-z)(b-z)}_{|z=b+(b-a)} = -\sqrt[+]{2}(b-a), \tag{2.4}$$

where $[a, b]$ denotes the interval $\{ta + (1-t)b | \; 0 \le t \le 1\}$ in $\mathbb{C}$. Using the product representation of $\Delta^2(\lambda, \varphi) - 4$ (cf. Proposition 2.4 (ii)), we now define, for $\lambda \in \mathbb{C}\setminus(\cup_{k\in\mathbb{Z}}[\lambda_k^-, \lambda_k^+])$, and $\varphi \in \mathcal{L}^2$, the canonical square root $\sqrt[c]{\Delta^2(\lambda, \varphi) - 4}$ by

$$\sqrt[c]{\Delta^2(\lambda, \varphi) - 4} := 2i\sqrt[s]{(\lambda_0^-(\varphi) - \lambda)(\lambda_0^+(\varphi) - \lambda)} \prod_{k\ne 0} \frac{\sqrt[s]{(\lambda_k^-(\varphi) - \lambda)(\lambda_k^+(\varphi) - \lambda)}}{k\pi}. \tag{2.5}$$

One easily sees that for any $\varphi \in \mathcal{L}^2_{\mathcal{R}}$ and $\lambda \in [\lambda_k^-(\varphi), \lambda_k^+(\varphi)] \subset \mathbb{R}$,

$$\pm(-1)^k \sqrt[c]{\Delta^2(\lambda \pm io, \varphi) - 4} > 0, \tag{2.6}$$

where o denotes a real positive infinitesimal increment.

γ-branch. Recall that for any $k \in \mathbb{Z}$ we denote by $(D_k)_{k\in\mathbb{Z}}$ the disk in $\mathbb{C}$ with center $k\pi$ and radius $\pi/4$.

Proposition 2.6. *There exists a neighborhood W of 0 in $\mathcal{L}^2$ so that for any $\varphi \in W \cap i\mathcal{L}^2_{\mathcal{R}}$, the following properties hold: For any $k \in \mathbb{Z}$ there exists a smooth arc $\gamma_k \subset D_k$ from $\lambda_k^-(\varphi)$ to $\lambda_k^+(\varphi)$ such that*

- (i) $\Delta(\lambda, \varphi) \in \mathbb{R}$, *for any* $\lambda \in \bigcup_{k\in\mathbb{Z}} \gamma_k$*;*
- (ii) *the orthogonal projection of γ_k to the imaginary axis is a diffeomorphism onto its image;*
- (iii) $\bar{\gamma}_k = \gamma_k$*;*
- (iv) $\dot{\lambda}_k \in \gamma_k \cap \mathbb{R}$*;*
- (v) $\Delta^2(\lambda, \varphi) - 4 < 0$ *for any* $\lambda \in \bigcup_{k\in\mathbb{Z}}(\gamma_k \backslash \{\lambda_k^+, \lambda_k^-\})$.

Remark. For related results for non-selfadjoint Hill's operators see also [12].

Proof. For any $\lambda \in \mathbb{C}$, write $\lambda = u + iv$ with $u, v \in \mathbb{R}$ and let $\Delta = \Delta_1 + i\Delta_2$, where $\Delta_1(u, v; \varphi) := \mathrm{Re}(\Delta(u + iv, \varphi))$ and $\Delta_2(u, v; \varphi) := \mathrm{Im}(\Delta(u + iv, \varphi))$. For any given $\varphi \in i\mathcal{L}^2_{\mathcal{R}}$ we want to study the zero level set of $\Delta_2(\lambda, \varphi) \equiv \Delta_2(u, v; \varphi)$ in $\mathbb{C}$. To this end, consider the function

$$F(u, v; \varphi) := \Delta_2(u, v; \varphi)/v\,. \tag{2.7}$$

By Proposition 2.4 (iii), $\Delta(\lambda, \varphi)$ is real-valued on $\mathbb{R} \times i\mathcal{L}^2_{\mathcal{R}}$. Hence, $\Delta_2(u, v; \varphi) = 0$ for $\lambda \in \mathbb{R}$, and thus, for any $\varphi \in i\mathcal{L}^2_{\mathcal{R}}$,

$$F(u, v; \varphi) = \int_0^1 (\partial_v \Delta_2)(u, vt; \varphi)\, dt\,. \tag{2.8}$$

As $\Delta(\lambda, \varphi)$ is an analytic function on $\mathbb{C}\times\mathcal{L}^2$, F is a real analytic function on $\mathbb{R}\times\mathbb{R}\times i\mathcal{L}^2_{\mathcal{R}}$, hence has an analytic extension to a neighborhood of $\mathbb{R} \times \mathbb{R} \times i\mathcal{L}^2_{\mathcal{R}}$ in $\mathbb{C} \times \mathbb{C} \times \mathcal{L}^2$ which we again denote by F. Note that for any given $\varphi \in i\mathcal{L}^2_{\mathcal{R}}$, the functions $F(\cdot, \cdot; \varphi)$ and $\Delta_2(\cdot, \cdot; \varphi)$ have the same zeroes in $\mathbb{R} \times (\mathbb{R}\backslash\{0\})$, hence it suffices to study the zero level sets of F. To this end consider the following map:

$$\mathcal{F} : B^\infty \times (-1, 1) \times i\mathcal{L}^2_{\mathcal{R}} \to l^\infty \equiv l^\infty(\mathbb{Z}, \mathbb{R}) \tag{2.9}$$

defined by $\mathcal{F} = (\mathcal{F}_k)_{k\in\mathbb{Z}}$ with

$$\mathcal{F}_k(\mathrm{u}, v; \varphi) := F(k\pi + u_k, v; \varphi), \quad \mathrm{u} := (u_k)_{k\in\mathbb{Z}}. \tag{2.10}$$

Here $B^\infty := \{\mathrm{u} \in l^\infty \mid \|\mathrm{u}\|_\infty < 1\}$.[1] It follows from (2.8), Cauchy's inequality (see e.g. Lemma A.2 in [8]) and Lemma I.2 in [6] that $\mathcal{F} : B^\infty \times (-1, 1) \times i\mathcal{L}^2_{\mathcal{R}} \to l^\infty$ extends to a locally bounded function $\mathcal{F}_{\mathbb{C}} : \mathcal{V}_{\mathbb{C}} \to l^\infty(\mathbb{Z}, \mathbb{C})$, where $\mathcal{V}_{\mathbb{C}}$ is an open neighborhood of $B^\infty \times (-1, 1) \times i\mathcal{L}^2_{\mathcal{R}}$ in $(B^\infty \times (-1, 1) \times i\mathcal{L}^2_{\mathcal{R}}) \otimes \mathbb{C}$. As for any $k \in \mathbb{Z}$

[1] More generally, for any $\delta > 0$ let $B^\infty_\delta := \{\mathrm{u} \in l^\infty \mid \|\mathrm{u}\|_\infty < \delta\}$.

the component $\mathcal{F}_k$ is analytic on $\mathcal{V}_{\mathbb{C}}$ (cf. (2.8)) we conclude from Theorem A.3 in [8] that $\mathcal{F}$ is real analytic on $B^\infty \times (-1, 1) \times i\mathcal{L}^2_{\mathcal{R}}$.

Note that $\Delta(\lambda, 0) = 2\cos\lambda$ and $\Delta_2(u, v; 0) = -2\sin u \sinh v$. Hence,

$$\mathcal{F}|_{\mathrm{u}=0, v=0, \varphi=0} = (-2\sin k\pi)_{k\in\mathbb{Z}} \equiv 0$$

and

$$\frac{\partial\mathcal{F}}{\partial \mathrm{u}}|_{\mathrm{u}=0, v=0, \varphi=0} = 2\,\mathrm{diag}((-1)^{k+1})_{k\in\mathbb{Z}}\,.$$

By the implicit function theorem there exist an open neighborhood W_1 of $\varphi = 0$ in $i\mathcal{L}^2_{\mathcal{R}}$, $\varepsilon > 0$, and a real analytic function

$$\mathcal{G} : (-\varepsilon, \varepsilon) \times W_1 \to l^\infty, \quad \mathcal{G} = (g_k)_{k\in\mathbb{Z}},$$

such that for any $v \in (-\varepsilon, \varepsilon)$ and any $\varphi \in W_1$,

$$\mathcal{F}(\mathcal{G}(v, \varphi), v; \varphi) = 0\,.$$

Moreover, there exists $\delta > 0$ such that the map $(-\varepsilon, \varepsilon) \times W_1 \to B^\infty \times (-\varepsilon, \varepsilon) \times W_1$,

$$(v, \varphi) \mapsto (\mathcal{G}(v, \varphi), v, \varphi)\,,$$

parametrizes the zero level set of $\mathcal{F}$ in $B^\infty_\delta \times (-\varepsilon, \varepsilon) \times W_1$. In particular, for any $\varphi \in W_1$ and any $k \in \mathbb{Z}$, the intersection of the zero level set of F with

$$D^\varepsilon_k := \{\lambda \in \mathbb{C} \mid |\,\mathrm{Re}(\lambda) - k\pi| < \delta, |\,\mathrm{Im}(\lambda)| < \varepsilon\}$$

is parametrized by $z_k : (-\varepsilon, \varepsilon) \to D^\varepsilon_k$,

$$v \mapsto k\pi + g_k(v, \varphi) + iv\,.$$

Let $\tilde{\gamma}_k := \mathrm{Image}(z_k) \subseteq D^\varepsilon_k$. By definition (2.7) of F, $\tilde{\gamma}_k\backslash\mathbb{R}$ coincides with the intersection of the zero level set of Δ_2 with $D^\varepsilon_k\backslash\mathbb{R}$. As $\Delta(\lambda, \varphi)$ is real for $\lambda \in \mathbb{R}$, we see that the intersection of the zero level set of Δ_2 with D^ε_k coincides with

$$Z_k := \tilde{\gamma}_k \cup (D^\varepsilon_k \cap \mathbb{R}) \subseteq \mathbb{C}\,.$$

Hence, for any $\varphi \in W_1$ and any $k \in \mathbb{Z}$, any complex number $\lambda \in D^\varepsilon_k$ satisfies

$$\Delta(\lambda, \varphi) \in \mathbb{R} \Longleftrightarrow \lambda \in Z_k\,. \tag{2.11}$$

Recall that at $\varphi = 0$, $\lambda^\pm_k = \dot{\lambda}_k = k\pi$ and $\Delta(\lambda^\pm_k) = 2(-1)^k$. Hence, by Proposition 2.1 and Proposition 2.4 (ii) there exists an open neighborhood W of $\varphi = 0$ in $\mathcal{L}^2$ such that $W \cap i\mathcal{L}^2_{\mathcal{R}} \subseteq W_1$ and for any $\varphi \in W \cap i\mathcal{L}^2_{\mathcal{R}}$ and any $k \in \mathbb{Z}$,

$$\lambda^\pm_k(\varphi),\ \dot{\lambda}_k(\varphi) \in D^\varepsilon_k \quad \text{and} \quad \Delta(\lambda^\pm_k(\varphi), \varphi) = 2(-1)^k\,.$$

Using that $\Delta(\lambda^\pm_k(\varphi), \varphi) = 2(-1)^k$ as well as the symmetry $\overline{\Delta(\bar{\lambda}, \varphi)} = \Delta(\lambda, \varphi)$ one sees that for any $\varphi \in W \cap i\mathcal{L}^2_{\mathcal{R}}$ and any $k \in \mathbb{Z}$,

$$\lambda^\pm_k(\varphi),\ \dot{\lambda}_k(\varphi) \in Z_k.$$

Now one easily sees that for any $\varphi \in W \cap i\mathcal{L}^2_{\mathcal{R}}$,

$$\gamma_k := \tilde{\gamma}_k \cap \{\lambda \in \mathbb{C} \mid |\Delta(\lambda)| \le 2\}$$

has the claimed properties. $\square$

Let W be a neighborhood of 0 in $\mathcal{L}^2$ as in Proposition 2.6. For φ in $W \cap i\mathcal{L}^2_{\mathcal{R}}$ we now define the following modification $\sqrt[\gamma_k]{(\lambda_k^-(\varphi) - \lambda)(\lambda_k^+(\varphi) - \lambda)}$ of the standard branch of the square root defined by (2.4): first define it for $\lambda \in \mathbb{C} \setminus D_k$ by

$$\sqrt[\gamma_k]{(\lambda_k^-(\varphi) - \lambda)(\lambda_k^+(\varphi) - \lambda)} := \sqrt[s]{(\lambda_k^-(\varphi) - \lambda)(\lambda_k^+(\varphi) - \lambda)}, \tag{2.12}$$

and then extend it by analyticity to $\mathbb{C} \setminus \gamma_k$. The γ-root of $\Delta^2(\lambda, \varphi) - 4$ in $\mathbb{C} \setminus \cup_{k \in \mathbb{Z}} \gamma_k$ is defined by

$$\sqrt[\gamma]{\Delta^2(\lambda, \varphi) - 4} := 2i \sqrt[\gamma_0]{(\lambda_0^-(\varphi) - \lambda)(\lambda_0^+(\varphi) - \lambda)} \prod_{k \neq 0} \frac{\sqrt[\gamma_k]{(\lambda_k^-(\varphi) - \lambda)(\lambda_k^+(\varphi) - \lambda)}}{k\pi}. \tag{2.13}$$

Similarly as for the canonical root of $\Delta^2(\lambda, \varphi) - 4$ for $\varphi \in \mathcal{L}^2_{\mathcal{R}}$, one verifies that for any $\varphi \in W \cap i\mathcal{L}^2_{\mathcal{R}}$, $k \in \mathbb{Z}$ and $\lambda \in \gamma_k$, we have

$$\pm (-1)^k i \sqrt[\gamma]{\Delta^2(\lambda \pm o, \varphi) - 4} > 0, \tag{2.14}$$

where o denotes a real positive infinitesimal increment.

2.3. Action variables for dNLS and their analytic extensions. Let $\varphi \in \mathcal{L}^2_{\mathcal{R}}$ be a potential of real type. Following [6], Sect. III.1, we associate to φ the k^{th} action variable

$$I_k(\varphi) := \frac{1}{\pi} \int_{\Gamma_k} \lambda \frac{\dot{\Delta}(\lambda, \varphi)}{\sqrt[c]{\Delta^2(\lambda, \varphi) - 4}} d\lambda, \tag{2.15}$$

where Γ_k is a counterclockwise oriented contour in $\mathbb{C}$ around the interval $[\lambda_k^-(\varphi), \lambda_k^+(\varphi)]$. The Γ_k are chosen so small that together with their interiors they do not intersect each other. Alternatively, I_k can be written as

$$I_k(\varphi) = \frac{1}{\pi} \int_{\Gamma_k} \log \left[(-1)^k \left(\Delta(\lambda, \varphi) - \sqrt[c]{\Delta^2(\lambda, \varphi) - 4} \right) \right] d\lambda. \tag{2.16}$$

By [6], Theorem III.2 and [6], Prop. III.21, we have the following results:

Proposition 2.7. *There exists a neighborhood W of $\mathcal{L}^2_{\mathcal{R}}$ in $\mathcal{L}^2$ such that for any $k \in \mathbb{Z}$, the action variable I_k analytically extends to potentials $\varphi \in W$ and*

(i) *(2.15)–(2.16) hold on W,*
(ii) *$\{I_j, I_k\} = 0$ for any $j, k \in \mathbb{Z}$.*

Proof. By Theorem III.2 in [6], I_k and I_j are real analytic functions on $\mathcal{L}^2_{\mathcal{R}}$. Hence by Proposition III.24 in [6], $\{I_k, I_j\}$ is real analytic as well and $\{I_k, I_j\}|_{\mathcal{L}^2_{\mathcal{R}}} = 0$. This shows that $\{I_k, I_j\} = 0$ in some neighborhood of $\mathcal{L}^2_{\mathcal{R}}$ in $\mathcal{L}^2$. $\square$

2.4. Angle variables for dNLS and their analytic extensions. Let $\varphi \in \mathcal{L}^2_{\mathcal{R}}$ and denote by $\Sigma(\varphi)$ the curve $\Sigma(\varphi) = \{(\lambda, z) : z^2 = \Delta^2(\varphi, \lambda) - 4\} \subset \mathbb{C}^2$. In view of definition (2.2), for any Dirichlet eigenvalue μ_k of $L(\varphi)$ one has

$$(M_{11} + M_{12})_{|1,\mu_k} = (M_{21} + M_{22})_{|1,\mu_k}. \tag{2.17}$$

Using (2.17) and the Wronskian identity $\det M(1, \lambda) = 1$, it follows that

$$\Delta^2(\mu_k, \varphi) - 4 = (M_{21} + M_{12})^2_{|1,\mu_k}.$$

The latter identity allows us to choose a sign of the root $\sqrt{\Delta^2(\mu_k, \varphi) - 4}$,

$$\sqrt[*]{\Delta^2(\mu_k, \varphi) - 4} := (M_{21} + M_{12})_{|1,\mu_k},$$

and hence the point μ_k^* on $\Sigma(\varphi)$

$$\mu_k^* = \left(\mu_k, \sqrt[*]{\Delta^2(\mu_k, \varphi) - 4}\right) := \left(\mu_k, (M_{21} + M_{12})_{|1,\mu_k}\right).$$

We refer to μ_k^* as a Dirichlet divisor. Following [6], Sect. III.3, we can associate to $\varphi \in \mathcal{L}^2_{\mathcal{R}}$ for any $k \in \mathbb{Z}$ with $\lambda_k^- < \lambda_k^+$, the k^{th} angle variable $\theta_k(\varphi)$, defined by the following path integral on $\Sigma(\varphi)$:

$$\theta_k(\varphi) := \sum_{j \in \mathbb{Z}} \int_{\lambda_j^-}^{\mu_j^*} \frac{\chi_k(\lambda)}{\sqrt{\Delta^2(\lambda) - 4}} d\lambda \bmod 2\pi, \tag{2.18}$$

where $\chi_n(\lambda) \equiv \chi_n(\lambda, \varphi)$, $n \in \mathbb{Z}$, is a family of analytic functions on $\mathbb{C} \times \mathcal{L}^2_{\mathcal{R}}$ uniquely determined by the normalization conditions

$$\frac{1}{2\pi} \int_{\Gamma_j} \frac{\chi_n(\lambda)}{\sqrt[c]{\Delta^2(\lambda) - 4}} d\lambda = \delta_{jn} \qquad \forall j, n \in \mathbb{Z}. \tag{2.19}$$

Each angle variable is real-analytic modulo 2π on the (dense) domain $\mathcal{L}^2_{\mathcal{R}} \setminus \mathcal{D}_k$, where

$$\mathcal{D}_k := \{\varphi \in \mathcal{L}^2 |\, \lambda_k^-(\varphi) = \lambda_k^+(\varphi)\}. \tag{2.20}$$

In fact, the right-hand side of (2.18), when taken modulo π, analytically extends to $W \setminus \mathcal{D}_k$, where W is a (sufficiently small) neighborhood of $\mathcal{L}^2_{\mathcal{R}}$ in $\mathcal{L}^2$ which is independent of k (cf. Theorem III.10 in [6]). By Theorem III.10, Proposition III.24, and Proposition III.25 in [6], the following results hold.

Proposition 2.8. *There exists a neighborhood W of $\mathcal{L}^2_{\mathcal{R}}$ in $\mathcal{L}^2$ so that for any $k \in \mathbb{Z}$, χ_k extends analytically to $\mathbb{C} \times W$ and θ_k, when taken modulo π, analytically extends to $W \setminus \mathcal{D}_k$, satisfying the following properties:*

(i) *relations* (2.18) *and* (2.19) *hold for any $k, n, j \in \mathbb{Z}$;*
(ii) $\left\{I_j, \theta_k\right\} = \delta_{jk}$ *on $W \setminus \mathcal{D}_k$ for any $j, k \in \mathbb{Z}$;*
(iii) $\left\{\theta_j, \theta_k\right\} = 0$ *on $W \setminus (\mathcal{D}_k \cup \mathcal{D}_j)$, for any $k, j \in \mathbb{Z}$.*

2.5. Birkhoff coordinates for dNLS and their analytic extensions. In [6], Chapt. III, it is shown that the map

$$\Phi : \mathcal{L}^2_{\mathcal{R}} \to l^2_{\mathbb{R}^2}, \quad \varphi \mapsto \Phi(\varphi) = (x_k(\varphi), y_k(\varphi))_{k\in\mathbb{Z}},$$

given by

$$(x_k(\varphi), y_k(\varphi)) = \begin{cases} \sqrt{2I_k(\varphi)}\,(\cos\theta_k(\varphi), \sin\theta_k(\varphi)) & \text{if } \varphi \in \mathcal{L}^2_{\mathcal{R}} \setminus \mathcal{D}_k \\ (0,0) & \text{if } \varphi \in \mathcal{L}^2_{\mathcal{R}} \cap \mathcal{D}_k \end{cases}$$

defines global Birkhoff coordinates. More precisely, the following theorem holds.

Theorem 2.1. *The map*

$$\Phi : \mathcal{L}^2_{\mathcal{R}} \to l^2_{\mathbb{R}^2}$$

is a diffeomorphism with the following properties:

(i) *Φ is bi-analytic and preserves the Poisson bracket.*
(ii) *The coordinates $(x_k, y_k)_{k\in\mathbb{Z}} = \Phi(\varphi)$ are Birkhoff coordinates for the defocusing NLS equation (and its hierarchy), i.e. for $\varphi \in \mathcal{L}^2_{\mathcal{R}} \cap (H^1 \times H^1)$, the push forward $\mathcal{H}_d \circ \Phi^{-1}$ of the dNLS-Hamiltonian $\mathcal{H}_d$ depends only on the action variables $I_k = \frac{1}{2}(x_k^2 + y_k^2)$, $k \in \mathbb{Z}$.*
(iii) *The differential at 0, $d_0\Phi : \mathcal{L}^2_{\mathcal{R}} \to l^2_{\mathbb{R}^2}$, is the Fourier transform (cf. [6], Prop. III.20). More precisely, for any $f = (f_1, f_2) \in \mathcal{L}^2_{\mathcal{R}}$, the image $(\xi, \eta) := d_0\Phi(f_1, f_2)$ is given by*

$$(\xi_k, \eta_k) = -\left(\frac{\hat f_1(-k) + \hat f_2(k)}{\sqrt{2}}, i\,\frac{\hat f_1(-k) - \hat f_2(k)}{\sqrt{2}} \right) \tag{2.21}$$

or

$$(\xi_k, \eta_k) = -(\sqrt{2}\,\mathrm{Re}\,\hat f_2(k), \sqrt{2}\,\mathrm{Im}\,\hat f_2(k)). \tag{2.22}$$

(iv) *For any $N \geq 1$, Φ maps $\mathcal{L}^2_{\mathcal{R}} \cap (H^N \times H^N)$ diffeomorphically onto $l^2_{\mathbb{R}^2} \cap (l^2_N \times l^2_N)$.*

By Theorem 2.1, the map $\Phi : \mathcal{L}^2_{\mathcal{R}} \to l^2_{\mathbb{R}^2}$ extends to an analytic map on a neighborhood of $\mathcal{L}^2_{\mathcal{R}}$ in $\mathcal{L}^2$ with values in $l^2_{\mathbb{C}^2}$:

Proposition 2.9. *There exists a neighborhood W of 0 in $\mathcal{L}^2$, and a neighborhood U of 0 in $l^2_{\mathbb{C}^2}$ such that Φ analytically extends to a map $W \to U$, which we again denote by Φ, satisfying the following properties:*

(i) *Φ is 1–1, onto, bi-analytic and preserves the Poisson bracket.*
(ii) *The push forward $\mathcal{H} \circ \Phi^{-1}$ of the Hamiltonian (1.3), restricted to $U \cap (l^2_1 \times l^2_1)$, depends only on the action variables $I_k = \frac{1}{2}(x_k^2 + y_k^2)$, $k \in \mathbb{Z}$.*
(iii) *The differential at 0, $d_0\Phi : \mathcal{L}^2 \to l^2_{\mathbb{C}^2}$ is the Fourier transform and is given by the formula (2.21) for arbitrary elements $(f_1, f_2) \in \mathcal{L}^2$.*
(iv) *For any $N \geq 0$, the restriction of Φ to $W \cap (H^N \times H^N)$ is a diffeomorphism $W \cap (H^N \times H^N) \to U \cap (l^2_N \times l^2_N)$.*

Proof. By Theorem 2.1,

$$(d_0\Phi)|_{\mathcal{L}^2_{\mathcal{R}}} = d_0(\Phi|_{\mathcal{L}^2_{\mathcal{R}}}) : \mathcal{L}^2_{\mathcal{R}} \to l^2_{\mathbb{R}^2}$$

is a linear $\mathbb{R}$-isomorphism given by formula (2.21). As Φ is real analytic it then follows that $d_0\Phi : \mathcal{L}^2 \to l^2_{\mathbb{C}^2}$ is a $\mathbb{C}$-linear isomorphism given by formula (2.21). The claimed statements then follow from the inverse function theorem and Theorem 2.1. □

3. Actions

In this section we want to show that the action variables for φ in a neighborhood of 0 in $i\mathcal{L}^2_{\mathcal{R}}$ are real valued. Let W be a neighborhood of 0 in $\mathcal{L}^2$ such that Proposition 2.5, Proposition 2.6, and Proposition 2.9 hold. The main result of this section is the following one.

Proposition 3.1. *For any $\varphi \in W \cap i\mathcal{L}^2_{\mathcal{R}}$, the action variables (2.15) are real valued.*

Proof. We have to show that for any $k \in \mathbb{Z}$, $I_k = \overline{I_k}$. By (2.15), and Proposition 2.7 (i),

$$I_k = \frac{1}{\pi}\int_{\Gamma_k} \lambda \frac{\dot{\Delta}(\lambda)}{\sqrt[c]{\Delta^2(\lambda)-4}}\, d\lambda, \tag{3.1}$$

where we chose Γ_k to be the (counterclockwise oriented) circle in $\mathbb{C}$ of center $k\pi$ and radius $\pi/4$. Then

$$\overline{I_k} = \frac{1}{\pi}\int_{\Gamma_k} \bar{\lambda} \frac{\overline{\dot{\Delta}(\lambda)}}{\overline{\sqrt[c]{\Delta^2(\lambda)-4}}}\, \overline{d\lambda}. \tag{3.2}$$

As $\lambda_k^- = \overline{\lambda_k^+}$ by Proposition 2.2, it follows from the definition of the standard branch of the square root (cf. Sect. 2.2), that

$$\sqrt[s]{(\lambda_k^- - \lambda)(\lambda_k^+ - \lambda)} = \overline{\sqrt[s]{(\lambda_k^- - \bar{\lambda})(\lambda_k^+ - \bar{\lambda})}},$$

and thus by (2.5),

$$\overline{\sqrt[c]{\Delta^2(\lambda)-4}} = -\sqrt[c]{\Delta^2(\bar{\lambda})-4}.$$

When combined with Proposition 2.4 (iii), formula (3.2) becomes

$$\bar{I}_k = \frac{1}{\pi}\int_{\Gamma_k} \bar{\lambda} \frac{\dot{\Delta}(\bar{\lambda})}{-\sqrt[c]{\Delta^2(\bar{\lambda})-4}}\, \overline{d\lambda}.$$

Parametrize Γ_k by $\lambda(t) = k\pi + \frac{\pi}{4}e^{it}$ with $0 \le t \le 2\pi$. Then $\overline{\lambda(t)} = \lambda(-t)$ and $\overline{d\lambda} = -\frac{i\pi}{4}e^{-it}dt$, and thus

$$\begin{aligned}
\overline{I_k} &= \frac{1}{\pi}\int_0^{2\pi} \lambda(-t) \frac{\dot{\Delta}(\lambda(-t))}{\sqrt[c]{\Delta^2(\lambda(-t))-4}}\, i\frac{\pi}{4}e^{i(-t)}dt \\
&= \frac{1}{\pi}\int_0^{2\pi} \lambda(s) \frac{\dot{\Delta}(\lambda(s))}{\sqrt[c]{\Delta^2(\lambda(s))-4}}\, i\frac{\pi}{4}e^{is}ds \\
&= I_k,
\end{aligned}$$

where for the latter identity we used again (3.1). □

In fact, one can show that for $\varphi \in W \cap i\mathcal{L}^2_{\mathcal{R}}$, the action variables are nonpositive.

Proposition 3.2. *Let W be the neighborhood of* 0 *in $\mathcal{L}^2$ as in Proposition 3.1. Then for any $k \in \mathbb{Z}$ and $\varphi \in W \cap i\mathcal{L}^2_{\mathcal{R}}$, we have $I_k \le 0$.*

Proof. It follows from Proposition 2.7 (i) that for any $k \in \mathbb{Z}$,

$$I_k(\varphi) = \frac{1}{\pi} \int_{\Gamma_k} \log \left[(-1)^k \left(\Delta(\lambda, \varphi) - \sqrt[c]{\Delta^2(\lambda, \varphi) - 4} \right) \right] d\lambda. \tag{3.3}$$

If $\lambda_k^- = \lambda_k^+$ then (3.3) shows that $I_k(\varphi) = 0$. So for the rest of the proof we assume that $\lambda_k^- \neq \lambda_k^+$. By (2.5), (2.12), and (2.13) one has $\sqrt[c]{\Delta^2(\lambda) - 4} = \sqrt[\gamma]{\Delta^2(\lambda) - 4}$ for $\lambda \in \bigcup_{k \in \mathbb{Z}} \Gamma_k$ and by (2.14), for any $k \in \mathbb{Z}$, $\lambda \in \gamma_k \setminus \left\{ \lambda_k^-, \lambda_k^+ \right\}$, and $\epsilon \in \{-1, +1\}$,

$$\epsilon (-1)^k i \sqrt[\gamma]{\Delta^2(\lambda + \epsilon o) - 4} > 0. \tag{3.4}$$

Hence, by Proposition 2.6 (v),

$$\epsilon (-1)^k i \sqrt[\gamma]{\Delta^2(\lambda + \epsilon o) - 4} = \sqrt[+]{4 - \Delta^2(\lambda)}, \tag{3.5}$$

where o denotes a real positive infinitesimal increment. In addition, it follows from (3.4) that for any $\lambda \in \bigcup_{k \in \mathbb{Z}} (\gamma_k \setminus \{\lambda_k^+, \lambda_k^-\})$ the imaginary part of $\sqrt[\gamma]{\Delta^2(\lambda \pm o) - 4}$ does not vanish. Hence, the sign of this imaginary part remains constant. As a consequence, for $\lambda \in \gamma_k \setminus \{\lambda_k^+, \lambda_k^-\}$, the principal branch of the logarithm

$$\log \left[(-1)^k \left(\Delta(\lambda) - \sqrt[\gamma]{\Delta^2(\lambda \pm o) - 4} \right) \right]$$

is well defined. By shrinking the contour Γ_k to γ_k, and assuming that γ_k is oriented, issuing from λ_k^- and ending at λ_k^+, we can write

$$\begin{aligned} I_k(\varphi) = {} & \frac{1}{\pi} \int_{\gamma_k} \log \left[(-1)^k \left(\Delta(\lambda) - \sqrt[\gamma]{\Delta^2(\lambda + o) - 4} \right) \right] d\lambda \\ & - \frac{1}{\pi} \int_{\gamma_k} \log \left[(-1)^k \left(\Delta(\lambda) - \sqrt[\gamma]{\Delta^2(\lambda - o) - 4} \right) \right] d\lambda. \end{aligned}$$

As by (3.5), for any $\epsilon \in \{-1, +1\}$,

$$(-1)^k \sqrt[\gamma]{\Delta^2(\lambda + \epsilon o) - 4} = -\epsilon\, i \sqrt[+]{4 - \Delta^2(\lambda)},$$

it then follows that

$$\begin{aligned} I_k(\varphi) = {} & \frac{1}{\pi} \int_{\gamma_k} \log \left[(-1)^k \Delta(\lambda) + i \sqrt[+]{4 - \Delta^2(\lambda)} \right] d\lambda \\ & - \frac{1}{\pi} \int_{\gamma_k} \log \left[(-1)^k \Delta(\lambda) - i \sqrt[+]{4 - \Delta^2(\lambda)} \right] d\lambda. \end{aligned} \tag{3.6}$$

Using that for $\lambda \in \gamma_k$,

$$\left| (-1)^k \Delta(\lambda) + i \sqrt[+]{4 - \Delta^2(\lambda)} \right| = \left| (-1)^k \Delta(\lambda) - i \sqrt[+]{4 - \Delta^2(\lambda)} \right|,$$

one sees that

$$\operatorname{Re} \left(\log \left[(-1)^k \Delta(\lambda) + i \sqrt[+]{4 - \Delta^2(\lambda)} \right] \right) = \operatorname{Re} \left(\log \left[(-1)^k \Delta(\lambda) - i \sqrt[+]{4 - \Delta^2(\lambda)} \right] \right).$$

Moreover, as $\Delta(\lambda)$ is real valued and $-2 \leq \Delta(\lambda) \leq 2$ for $\lambda \in \gamma_k$, one has

$$\mathrm{Im}\left(\log\left[(-1)^k\Delta(\lambda) + i\sqrt[+]{4-\Delta^2(\lambda)}\right]\right) = -\mathrm{Im}\left(\log\left[(-1)^k\Delta(\lambda) - i\sqrt[+]{4-\Delta^2(\lambda)}\right]\right).$$

Hence (3.6) leads to the identity

$$I_k(\varphi) = \frac{2}{\pi}\int_{\gamma_k} i\ \mathrm{Im}\left(\log\left[(-1)^k\Delta(\lambda) + i\sqrt[+]{4-\Delta^2(\lambda)}\right]\right) d\lambda.$$

To evaluate the latter integral, parametrize the path γ_k by the imaginary part. By Proposition 2.6 (ii) this is possible, i.e. there exists a C^1-curve $t \mapsto a(t)$ so that

$$\lambda(t) = a(t) + ti, \quad |t| \leq \mathrm{Im}\lambda_k^+.$$

Then, with $\dot{a}(t) = \frac{d}{dt}a(t)$,

$$d\lambda = (\dot{a} + i)dt.$$

As the action variables are real valued by Proposition 3.1, we get

$$I_k(\varphi) = -\frac{2}{\pi}\int_{\mathrm{Im}\lambda_k^-}^{\mathrm{Im}\lambda_k^+} \mathrm{Im}\left(\log\left[(-1)^k\Delta(\lambda(t)) + i\sqrt[+]{4-\Delta^2(\lambda(t))}\right]\right) dt.$$

Since for any $|t| < \mathrm{Im}\lambda_k^+$,

$$\mathrm{Im}\left((-1)^k\Delta(\lambda(t)) + i\sqrt[+]{4-\Delta^2(\lambda(t))}\right) = \sqrt[+]{4-\Delta^2(\lambda(t))} > 0,$$

one concludes that

$$\mathrm{Im}\left(\log\left[(-1)^k\Delta(\lambda(t)) + i\sqrt[+]{4-\Delta^2(\lambda(t))}\right]\right) \in\ (0,\pi).$$

Thus we have shown that $I_k(\varphi) < 0$ for any $k \in \mathbb{Z}$ with $\lambda_k^- \neq \lambda_k^+$. $\square$

For $\varphi \in W \cap i\mathcal{L}^2_{\mathcal{R}}$, Proposition 3.2 can be used to obtain a formula for the Birkhoff coordinates $(x_k, y_k)_{k\in\mathbb{Z}}$ provided by Proposition 2.9. It follows from the construction in [6], III.4, that for any $\varphi \in W\backslash\mathcal{D}_k$,

$$x_k = \sqrt{2}\,\xi_k\frac{\lambda_k^+ - \lambda_k^-}{2}\cos\theta_k \quad \text{and} \quad y_k = \sqrt{2}\,\xi_k\frac{\lambda_k^+ - \lambda_k^-}{2}\sin\theta_k, \tag{3.7}$$

where θ_k is defined by formula (2.18) and where

$$\xi_k := \sqrt[+]{4I_k/(\lambda_k^+ - \lambda_k^-)^2} \tag{3.8}$$

is a real analytic non-vanishing function defined on W (cf. Theorem III.3 in [6]). On $W\backslash\mathcal{D}_k$, the angle variable θ_k is analytic modulo π. When taken modulo 2π, θ_k might not be continuous. In fact, continuous deformations of φ in $W\backslash\mathcal{D}_k$ could lead to discontinuities of λ_k^- and λ_k^+ due to the imposed lexicographic ordering $\lambda_k^- \preccurlyeq \lambda_k^+$, and hence to an increment $\pm\pi$ on the right-hand side of (2.18). It follows from Proposition 2.2 (ii) and Proposition 2.5 (i) that for continuous deformations of φ in the smaller set $(W \cap i\mathcal{L}^2_{\mathcal{R}})\backslash\mathcal{D}_k$, the eigenvalues $\lambda_k^- = \bar{\lambda}_k^+$ and λ_k^+ change continuously. Hence, for $\varphi \in (W \cap i\mathcal{L}^2_{\mathcal{R}})\backslash\mathcal{D}_k$ the angle variable

$$\theta_k(\varphi) = \sum_{j\in\mathbb{Z}} \int_{\lambda_j^-}^{\mu_j^*} \frac{\chi_k(\lambda, \varphi)}{\sqrt{\Delta^2(\lambda, \varphi) - 4}}\, d\lambda \bmod 2\pi \tag{3.9}$$

is an analytic function on $(W \cap i\mathcal{L}^2_{\mathcal{R}})\backslash\mathcal{D}_k$ and by (3.7), (3.8), and Proposition 3.2 we get that

$$x_k = i\sqrt[+]{-2I_k}\cos\theta_k \quad \text{and} \quad y_k = i\sqrt[+]{-2I_k}\sin\theta_k \tag{3.10}$$

for any $\varphi \in i\mathcal{L}^2_{\mathcal{R}}\backslash\mathcal{D}_k$.

4. Even Potentials

To prove Theorem 1.1, the notion of even potentials will play an important role. In this section, we assume that W is a neighborhood of 0 in $\mathcal{L}^2$, chosen in such a way that Propositions 2.6, 2.7, 2.8, 2.9 and Propositions 3.1, 3.2 hold. Denote by U the image of W by the bi-analytic map Φ of Proposition 2.9, $U = \Phi(W)$.

Definition 4.1. *An element $\varphi = (\varphi_1, \varphi_2)$ in $\mathcal{L}^2$ is said to be even if*

$$\varphi_2(x) = \varphi_1(1 - x) \quad a.e.\ x \in \mathbb{R}.$$

Note that $\varphi = (\psi, \bar{\psi}) \in \mathcal{L}^2_{\mathcal{R}}$ is even iff $\psi(x) = \bar{\psi}(1 - x)$ a.e. whereas $\varphi = (\psi, -\bar{\psi}) \in i\mathcal{L}^2_{\mathcal{R}}$ is even iff $\psi(x) = -\bar{\psi}(1 - x)$ a.e. . Denote by $\mathcal{L}^2_{\mathcal{R},\text{ even}}$ [$\mathcal{L}^2_{\text{even}}$] the set of even potentials in $\mathcal{L}^2_{\mathcal{R}}$ [$\mathcal{L}^2$]. Then $i\mathcal{L}^2_{\mathcal{R},\text{ even}}$ is the set of even potentials in $i\mathcal{L}^2_{\mathcal{R}}$.

Definition 4.2. *An element $(x, y) = (x_k, y_k)_{k\in\mathbb{Z}}$ in $l^2_{\mathbb{C}^2}$ is said to be even iff $y_k = 0$ for any $k \in \mathbb{Z}$.*

Denote by $l^2_{\mathbb{R}^2,\text{ even}}$ the even elements of $l^2_{\mathbb{R}^2}$. Then $il^2_{\mathbb{R}^2,\text{ even}}$ is the set of even elements of $il^2_{\mathbb{R}^2}$.

Lemma 4.1. *$d_0\Phi|_{i\mathcal{L}^2_{\mathcal{R},\text{ even}}} : i\mathcal{L}^2_{\mathcal{R},\text{ even}} \to il^2_{\mathbb{R}^2,\text{ even}}$ is a $\mathbb{R}$-linear isomorphism.*

Proof. The claimed statement follows easily from formula (2.21) of Theorem 2.1. $\square$

Next we want to show that $\Phi(W \cap i\mathcal{L}^2_{\mathcal{R},\text{ even}}) \subseteq il^2_{\mathbb{R}^2,\text{ even}}$. For this we first need to establish a few auxiliary results.

Lemma 4.2. *For any $k \in \mathbb{Z}$, $(W \cap i\mathcal{L}^2_{\mathcal{R},\text{ even}})\backslash\mathcal{D}_k$ is dense in $W \cap i\mathcal{L}^2_{\mathcal{R},\text{ even}}$.*

Proof. First note that by formula (2.15), $W \cap \mathcal{D}_k$ is contained in the zero set of the action variable I_k, i.e. $W \cap \mathcal{D}_k \subseteq \{\varphi \in W \mid I_k(\varphi) = 0\}$. Assume that the claimed statement does *not* hold. Then there exists $k \in \mathbb{Z}$ and a non empty, open set $U \subseteq W$ so that I_k vanishes on $U \cap i\mathcal{L}^2_{\mathcal{R},\text{ even}}$. Note that $\mathcal{L}^2_{\text{even}} = (i\mathcal{L}^2_{\mathcal{R},\text{ even}}) \otimes \mathbb{C}$ and recall that I_k is real-valued on $W \cap i\mathcal{L}^2_{\mathcal{R},\text{ even}}$. It then follows from $I_k|_{U\cap i\mathcal{L}^2_{\mathcal{R},\text{ even}}} \equiv 0$ that $I_k \equiv 0$ on a non empty connected component of $W \cap \mathcal{L}^2_{\text{even}}$, contradicting Theorem 2.1. $\square$

Lemma 4.3. *For any* $\varphi \in W \cap \mathcal{L}^2_{even}$, $\mu_k(\varphi) \in \{\lambda_k^+(\varphi), \lambda_k^-(\varphi)\}$ $\forall k \in \mathbb{Z}$.

Proof. Let $\varphi = (\varphi_1, \varphi_2) \in \mathcal{L}^2 \cap W$ be an even potential. Let $F = (F_1, F_2)$ be a Dirichlet eigenfunction associated to the kth Dirichlet eigenvalue $\mu_k(\varphi)$, i.e.

$$\begin{cases} i\partial_x F_1 + \varphi_1 F_2 &= \mu_k F_1 \\ -i\partial_x F_2 + \varphi_2 F_1 &= \mu_k F_2 \end{cases} \tag{4.1}$$

and $F_1(0) = F_2(0)$, $F_1(1) = F_2(1)$. Let $\tilde{F}(x) := (F_2(1-x), F_1(1-x))$. Note that $\tilde{F}$ satisfies the same boundary conditions as F. To see that $\tilde{F}$ is a solution of (4.1), interchange the two equations in (4.1) and evaluate them at $1-x$. As $(\partial_x F_j)(1-x) = -\partial_x(F_j(1-x))$, one gets

$$\begin{cases} i\partial_x\,(F_2(1-x)) + \varphi_2(1-x)F_1(1-x) &= \mu_k F_2(1-x) \\ -i\partial_x(F_1(1-x)) + \varphi_1(1-x)F_2(1-x) &= \mu_k F_1(1-x). \end{cases}$$

Using the assumption that φ is even, one then concludes that

$$\begin{cases} i\partial_x \tilde{F}_1(x) + \varphi_1(x)\tilde{F}_2(x) &= \mu_k \tilde{F}_1(x) \\ -i\partial_x \tilde{F}_2(x) + \varphi_2(x)\tilde{F}_1(x) &= \mu_k \tilde{F}_2(x). \end{cases}$$

Hence $\tilde{F}$ is an eigenfunction for the Dirichlet eigenvalue μ_k. We now distinguish between two cases: If $\tilde{F} = F$, then

$$(F_1(0), F_2(0)) = (F_2(1), F_1(1)).$$

Since F satisfies Dirichlet boundary conditions, $F_1(0) = F_2(0)$ and $F_1(1) = F_2(1)$, it satisfies periodic boundary conditions as well. If $\tilde{F} \neq F$, then $F - \tilde{F}$ is a non-trivial solution of the system (4.1), which satisfies anti-periodic boundary conditions, i.e.

$$(F - \tilde{F})(1) = -(F - \tilde{F})(0).$$

In other words we have shown that $\mu_k(\varphi) \in \left\{\lambda_j^{\pm}(\varphi),\ j \in \mathbb{Z}\right\}$. Lemma 4.3 then follows from Proposition 2.5 (i) and (iii). □

Lemma 4.4. $\Phi(W \cap i\mathcal{L}^2_{\mathcal{R},\, even}) \subseteq il^2_{\mathbb{R}^2,\, even}$.

Proof. Let $\varphi \in W \cap i\mathcal{L}^2_{\mathcal{R},\,\text{even}}$. By Proposition 2.8 (i), for any $k \in \mathbb{Z}$ with $\lambda_k^+(\varphi) \neq \lambda_k^-(\varphi)$, the angle variable $\theta_k(\varphi)$ is well defined by (2.18) and the normalizing condition (2.19)

$$\int_{\Gamma_j} \frac{\chi_k(\lambda)}{\sqrt[c]{\Delta^2(\lambda) - 4}} d\lambda = 2\pi\delta_{jk}$$

is valid. Shrink the contour Γ_j to the arc γ_j, given by Proposition 2.6, to get, in view of formula (2.19) and Proposition 2.8 (i),

$$\int_{\gamma_j} \frac{\chi_k(\lambda)}{\sqrt{\Delta^2(\lambda) - 4}} d\lambda \in \left\{\pm\pi\delta_{jk}\right\}.$$

By Lemma 4.3, $\mu_k \in \{\lambda_k^-(\varphi), \lambda_k^+(\varphi)\}$ for any $k \in \mathbb{Z}$. Hence for any $k \in \mathbb{Z}$ with $\lambda_k^+(\varphi) \neq \lambda_k^-(\varphi)$,

$$\theta_k(\varphi) = \sum_{j\in\mathbb{Z}} \int_{\lambda_j^-}^{\mu_j^*} \frac{\chi_k(\lambda)}{\sqrt{\Delta^2(\lambda)-4}} d\lambda \in \{0, \pi\} \mod 2\pi.$$

By formula (3.10) for $(x, y) = \Phi(\varphi)$ it then follows that for such k's,

$$x_k(\varphi) = i\sqrt[+]{-2I_k(\varphi)} \quad \text{and} \quad y_k(\varphi) = i\sqrt[+]{-2I_k(\varphi)} \sin\theta_k(\varphi) = 0$$

on $(W \cap i\mathcal{L}^2_{\mathcal{R},\,\text{even}}) \setminus \mathcal{D}_k$. It then follows by Proposition 3.2 that $x_k \in i\mathbb{R}$. By Lemma 4.2, $(W \cap i\mathcal{L}^2_{\mathcal{R},\,\text{even}}) \setminus \mathcal{D}_k$ is dense in $W \cap i\mathcal{L}^2_{\mathcal{R},\,\text{even}}$. By the continuity of x_k and y_k it then follows that $x_k \in i\mathbb{R}$ and $y_k = 0$ on $W \cap i\mathcal{L}^2_{\mathcal{R},\,\text{even}}$. This shows that

$$\Phi(W \cap i\mathcal{L}^2_{\mathcal{R},\,\text{even}}) \subseteq il^2_{\mathbb{R}^2,\,\text{even}}$$

as claimed. □

Proposition 4.1. *By shrinking W and U if necessary, it follows that*

$$\Phi|_{W\cap i\mathcal{L}^2_{\mathcal{R},\,even}} : W \cap i\mathcal{L}^2_{\mathcal{R},\,even} \to U \cap il^2_{\mathbb{R}^2,\,even}$$

is a diffeomorphism.

Proof. In view of Lemma 4.1 and Lemma 4.4 the claimed statement follows from the inverse function theorem. □

5. The Real Symplectic Subspace $i\mathcal{L}^2_{\mathcal{R}}$

Recall that we have introduced the real subspace $\mathcal{L}^2_{\mathcal{R}}$ of $\mathcal{L}^2 = L^2(\mathbb{T}, \mathbb{C}) \times L^2(\mathbb{T}, \mathbb{C})$ given by

$$\mathcal{L}^2_{\mathcal{R}} = \left\{ \varphi = (\psi, \bar{\psi}) | \ \psi \in L^2(\mathbb{T}, \mathbb{C}) \right\}.$$

Note that $i\mathcal{L}^2_{\mathcal{R}}$ is a real subspace of $\mathcal{L}^2$ as well and for any $\varphi = (\varphi_1, \varphi_2) \in \mathcal{L}^2$ one has

$$\varphi \in i\mathcal{L}^2_{\mathcal{R}} \quad \text{iff} \quad \varphi_2 = -\bar{\varphi}_1. \tag{5.1}$$

The subspace $i\mathcal{L}^2_{\mathcal{R}}$ can be identified with $L^2(\mathbb{T}, \mathbb{R}) \times L^2(\mathbb{T}, \mathbb{R})$ in a natural way. To this end introduce the $\mathbb{C}$-linear isomorphism $T : L^2(\mathbb{T}, \mathbb{C}) \times L^2(\mathbb{T}, \mathbb{C}) \to \mathcal{L}^2$,

$$(\psi_1, \psi_2) \mapsto (\varphi_1, \varphi_2) = \frac{1}{\sqrt{2}}(\psi_1 + i\psi_2, -\psi_1 + i\psi_2).$$

In a straightforward way one shows the following lemma.

Lemma 5.1. (i) *$i\mathcal{L}^2_{\mathcal{R}}$ is the image by T of the real subspace $L^2(\mathbb{T}, \mathbb{R}) \times L^2(\mathbb{T}, \mathbb{R})$ of $L^2(\mathbb{T}, \mathbb{C}) \times L^2(\mathbb{T}, \mathbb{C})$.*

(ii) *T is canonical when its domain of definition is endowed with the canonical Poisson structure*

$$\{F, G\}_0 (\psi_1, \psi_2) = \int_0^1 \left(\partial_{\psi_1} F \, \partial_{\psi_2} G - \partial_{\psi_2} F \, \partial_{\psi_1} G\right) dx$$

and the target of T with the Poisson bracket introduced in Sect. 1.

Now consider an analytic functional $F : W \to \mathbb{C}$ defined in a neighborhood W of 0 in $\mathcal{L}^2$.

Lemma 5.2. *If $F|_{i\mathcal{L}^2_{\mathcal{R}}}$ is real valued then the Hamiltonian vector field $(-i\partial_{\varphi_2} F, i\partial_{\varphi_1} F)$ is tangent to $i\mathcal{L}^2_{\mathcal{R}}$.*

Proof. Consider the pull back $F \circ T$ of F. Then $F \circ T$ is an analytic functional on $W' := T^{-1}(W)$ whose restriction to $\left(L^2(\mathbb{T}, \mathbb{R}) \times L^2(\mathbb{T}, \mathbb{R})\right) \cap W'$ is real valued. This implies that the Hamiltonian vector field $(-\partial_{\psi_2}(F \circ T), \partial_{\psi_1}(F \circ T))$ takes values in $L^2(\mathbb{T}, \mathbb{R}) \times L^2(\mathbb{T}, \mathbb{R})$ on $W' \cap \left(L^2(\mathbb{T}, \mathbb{R}) \times L^2(\mathbb{T}, \mathbb{R})\right)$. As T is canonical,

$$T\left(-\partial_{\psi_2}(F \circ T), \partial_{\psi_1}(F \circ T)\right) = (-i\partial_{\varphi_2} F, i\partial_{\varphi_1} F),$$

and this vector field is tangent to $i\mathcal{L}^2_{\mathcal{R}}$ on $W \cap i\mathcal{L}^2_{\mathcal{R}}$ as T maps $L^2(\mathbb{T}, \mathbb{R}) \times L^2(\mathbb{T}, \mathbb{R})$ to $i\mathcal{L}^2_{\mathcal{R}}$. □

6. Proof of Theorem 1.1

The idea of our proof can be best explained in terms of the Birkhoff coordinates $(x_k, y_k)_{k\in\mathbb{Z}}$. We consider the sequence of Hamiltonian vector fields

$$X^{(k)}(x, y) := ((-y_k, x_k)\delta_{kl})_{l\in\mathbb{Z}}$$

on $il^2_{\mathbb{R}^2}$ with Hamiltonian $I_k = \frac{1}{2}(x_k^2 + y_k^2)$ and study their integral curves. For any $k \in \mathbb{Z}$, the solution $\left(x_l^{(k)}(t), y_l^{(k)}(t)\right)_{l\in\mathbb{Z}}$ of the initial value problem

$$(\dot{x}_l, \dot{y}_l) = (-y_k, x_k)\,\delta_{kl} \quad \forall\, l \in \mathbb{Z}, \tag{6.1}$$

$$(x_l(0), y_l(0)) = (\xi_l, \eta_l) \in i\mathbb{R}^2 \quad \forall\, l \in \mathbb{Z}, \tag{6.2}$$

is given by

$$\left(x_l^{(k)}(t), y_l^{(k)}(t)\right) = \begin{cases} (\xi_l, \eta_l) & \forall\, l \neq k \\ (\xi_k \cos t - \eta_k \sin t,\ \xi_k \sin t + \eta_k \cos t) & l = k. \end{cases}$$

Clearly, it exists for all time and evolves in $il^2_{\mathbb{R}^2}$. Actually, it evolves in $\mathrm{Iso}(\xi, \eta) \cap il^2_{\mathbb{R}^2}$, where for $(x, y) = (x_k, y_k)_{k\in\mathbb{Z}}$ in $l^2_{\mathbb{C}^2}$ we denote by $\mathrm{Iso}(x, y)$ the set of sequences

$$\mathrm{Iso}(x, y) := \left\{(x_k', y_k')_{k\in\mathbb{Z}} \in l^2_{\mathbb{C}^2} \,|\, x_k'^2 + y_k'^2 = x_k^2 + y_k^2\ \forall k \in \mathbb{Z}\right\}.$$

We want to show that any given point in $\mathrm{Iso}(x, y) \cap il^2_{\mathbb{R}^2}$ can be reached from any other point in $\mathrm{Iso}(x, y) \cap il^2_{\mathbb{R}^2}$ by concatenating integral curves of the above vector fields. First we follow the integral curve of $X^{(0)}$ which starts at the point $(\xi, 0)$, where $\xi = (\xi_l)_{l\in\mathbb{Z}} \in il^2(\mathbb{Z}, \mathbb{R})$ is given by

$$\xi_l = i\sqrt[+]{|x_l|^2 + |y_l|^2} \quad \forall l \in \mathbb{Z} \tag{6.3}$$

until we reach the point $(\xi^{(0)}, \eta^{(0)})$ where

$$(\xi_l^{(0)}, \eta_l^{(0)}) = \begin{cases} (\xi_l, 0) & \text{if } l \neq 0 \\ (x_0, y_0) & \text{if } l = 0. \end{cases}$$

Then we continue on the integral curve of $X^{(1)}$ until we reach $(\xi^{(1)}, \eta^{(1)})$ where

$$(\xi_l^{(1)}, \eta_l^{(1)}) = \begin{cases} (\xi_l^{(0)}, \eta_l^{(0)}) & \text{if } l \neq 1 \\ (x_1, y_1) & \text{if } l = 1. \end{cases}$$

Next we continue on the integral curve of $X^{(-1)}$ until we have reached $(\xi^{(-1)}, \eta^{(-1)})$ where

$$(\xi_l^{(-1)}, \eta_l^{(-1)}) = \begin{cases} (\xi_l^{(1)}, \eta_l^{(1)}) & \text{if } l \neq -1 \\ (x_{-1}, y_{-1}) & \text{if } l = -1. \end{cases}$$

In this way we construct a sequence of points in $\mathrm{Iso}(x, y) \cap il^2_{\mathbb{R}^2}$,

$$(\xi, 0),\ (\zeta^{(0)}, \eta^{(0)}),\ (\xi^{(1)}, \eta^{(1)}),\ (\xi^{(-1)}, \eta^{(-1)}), \ldots. \tag{6.4}$$

It is easy to see that this sequence converges to the point (x, y). In order to prove Theorem 1.1 we apply Φ^{-1} to such sequences of points and use Proposition 4.1 and Lemma 5.2 to conclude that their images are in $i\mathcal{L}^2_{\mathcal{R}}$.

Proof of Theorem 1.1. By Proposition 2.9 there exist an open neighborhood W of 0 in $\mathcal{L}^2$, an open neighborhood U of 0 in $l^2_{\mathbb{C}^2}$, and a diffeomorphism $\Phi : W \to U$ so that $\Phi(W \cap i\mathcal{L}^2_{\mathcal{R}}) = U \cap il^2_{\mathbb{R}^2}$. By Proposition 4.1 we can assume that

$$\Phi(W \cap i\mathcal{L}^2_{\mathcal{R},\,\text{even}}) = U \cap il^2_{\mathbb{R}^2,\,\text{even}}.$$

Without loss of generality we may assume that U is a ball. In a first step we want to prove that

$$\Phi^{-1}(U \cap il^2_{\mathbb{R}^2}) \subseteq W \cap i\mathcal{L}^2_{\mathcal{R}}.$$

Let (x, y) be an arbitrary point in $U \cap il^2_{\mathbb{R}^2}$. As U is assumed to be a ball it follows that $(\xi, 0)$, defined by (6.3), is also in U, hence

$$(\xi, 0) \in U \cap il^2_{\mathbb{R}^2,\,\text{even}}.$$

By Proposition 4.1 it follows that $\zeta := \Phi^{-1}(i\xi, 0)$ is in $W \cap i\mathcal{L}^2_{\mathcal{R}}$. As Φ is canonical, the pull backs of the vector fields $X^{(k)}$ by Φ are again Hamiltonian vector fields. They are given by $(k \in \mathbb{Z})$,

$$Y^{(k)} = i(-\partial_{\varphi_2} I_k, \partial_{\varphi_1} I_k).$$

We recall that I_k are analytic functionals on W which are real valued on $i\mathcal{L}^2_{\mathcal{R}}$. Hence by Lemma 5.2, the vector fields $Y^{(k)}$ when restricted to $W \cap i\mathcal{L}^2_{\mathcal{R}}$ are tangent to $i\mathcal{L}^2_{\mathcal{R}}$. It then follows that the sequence

$$\zeta^{(k)} := \Phi^{-1}(\xi^{(k)}, \eta^{(k)})$$

is in $i\mathcal{L}^2_{\mathcal{R}}$ where $(\xi^{(k)}, \eta^{(k)})$ is given by (6.4). As $i\mathcal{L}^2_{\mathcal{R}}$ is closed in $\mathcal{L}^2$ and Φ is continuous one concludes that

$$\lim_{k\to\infty} \zeta^{(k)} = \lim_{k\to\infty} \Phi^{-1}(\xi^{(k)}, \eta^{(k)}) = \Phi^{-1}(x, y)$$

is an element in $i\mathcal{L}^2_{\mathcal{R}}$. This shows that

$$\Phi^{-1}(U \cap il^2_{\mathbb{R}^2}) \subseteq W \cap i\mathcal{L}^2_{\mathcal{R}}. \tag{6.5}$$

By Proposition 2.9 (iii), the differential of Φ at 0, $d_0\Phi : \mathcal{L}^2 \to l^2_{\mathbb{C}^2}$, is a $\mathbb{C}$-linear isomorphism. By applying the inverse function theorem and using (6.5) once more one then concludes that there exists a neighborhood $U_f \subseteq U \cap il^2_{\mathbb{R}^2}$ of 0 in $il^2_{\mathbb{R}^2}$ and a neighborhood $W_f \subseteq W \cap i\mathcal{L}^2_{\mathcal{R}}$ of 0 in $i\mathcal{L}^2_{\mathcal{R}}$ so that

$$\Phi : W_f \to U_f$$

is a diffeomorphism. The properties of $\Phi_f := \Phi|_{W_f}$, stated in items $(i) - (iii)$ of Theorem 1.1, now follow from the corresponding properties of the Birkhoff map $\Phi : W \to U$ (Proposition 2.9, items (i),(ii), and (iv)) in a straightforward way. $\square$

References

1. Ablowitz, M., Ma, Y.: The periodic cubic Schrödinger equation. Studies Appl. Math. **65**, 113–158 (1981)
2. Bambusi, D., Grébert, B.: Birkhoff normal form for PDE's with tame modulus. Duke Math. J. **135**, 507–567 (2005)
3. Grébert, B., Guillot, J.C.: Gaps of one-dimensional periodic AKNS systems. Forum Math. **5**, 459–504 (1993)
4. Grébert, B., Kappeler, T.: Symmetries of the Nonlinear Schrödinger equation. Bull. Soc. Math. France **130**(4), 603–618 (2002)
5. Grébert, B., Kappeler, T.: Perturbations of the defocusing nonlinear Schrödinger equation. Milan J. Math. **71**, 141–174 (2003)
6. Grébert, B., Kappeler, T., Pöschel, J.: *Normal form theory for the nonlinear Schrödinger equation*. Preliminary version available at http://www.math.sciences.univ-nantes.fr/~grebert/publication.html, 2002
7. Ito, H.: Convergent normal forms for integrable systems. Comment. Math. Helv. **64**, 412–461 (1989)
8. Kappeler, T., Pöschel, J.: *KdV & KAM*. Ergeb. der Math. und ihrer Grenzgeb., Berlin-Heidelberg-New York: Springer Verlag, 2003
9. Kappeler, T., Serier, F., Topalov, P.: *On the characterisation of the smoothness of skew-adjoint potentials in periodic Dirac operators*. Preprint
10. Kuksin, S.B., Pöschel, J.: Invariant Cantor manifolds of quasi-periodic oscillations for a nonlinear Schrödinger equation. Ann. Math. **143**, 149–179 (1996)
11. Li, Y., McLaughlin, D.W.: Morse and Melnikov functions for NLS Pde's. Commun. Math. Phys. **162**, 175–214 (1994)
12. Tkachenko, V.A.: Spectra of non-selfadjoint Hill's operators and a class of Riemann surfaces. Ann. Math. **143**, 181–231 (1996)
13. Vey, J.: Sur certains systemes dynamiques separables. Amer. J. Math. **100**, 591–614 (1978)
14. Zakharov, V.E., Shabat, A.B.: A scheme for integrating nonlinear equations of mathematical physics by the method of the inverse scattering problem I. Funct. Anal. Appl. **8**, 226–235 (1974)
15. Zung, N.T.: Convergence versus integrability in Birkhoff normal forms. Ann. Math. **161**, 141–156 (2005)

Communicated by G. Gallavotti